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Time-Closed Optimal Transfer by Two Impulses Between Coplanar Elliptical Orbits

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Introduction

GIVEN a pair of coplanar elliptical orbits about a center of gravitational attraction, on the assumption that a rocket vehicle is to be transferred from one to the other using two impulsive thrusts, equations governing the transfer orbit corresponding to minimum propellant expenditure have been found[1,2] and possible solutions discussed.[3] The analysis assumed that the transfer time was not prescribed, i.e., the time-open case. The time-closed problem has been studied by a number of researchers,[4-7] but the amendments that need to be made to the optimizing equations in this case have not been calculated. The required additional terms are derived in this Note.

The statement of the time-open problem does not require the identification of terminals on the orbits of departure and arrival. For the time-closed problem, however, the prescribed transit time T must be measured between a fixed terminal A on the departure orbit and another fixed terminal B on the arrival orbit. We shall assume that, for minimum propellant expenditure, the rocket coasts for a time t_1 from A along the departure orbit to a junction point J at which the first impulsive thrust is applied, that it then coasts for a time τ along the transfer orbit to a junction point K on the arrival orbit, at which a second impulsive thrust is given, and that it then finally coasts for a time t_2 to B along the arrival orbit. Thus, the closed-time constraint is

$$t_1 + \tau + t_2 = T \tag{1}$$

We shall, however, permit the rocket to leave the departure orbit prior to its arrival at A and to enter the arrival orbit at a point subsequent to B, provided the synchrony of its final motion with that prescribed is still guaranteed. Mathematically, this relaxation permits t_1 and t_2 to take negative values in the constraint (1).

Preliminary Equations

Each elliptical orbit will be specified by the values of its elements, i.e., l, the semilatus rectum; e, the eccentricity; and ω, the longitude of the periapse measured from any convenient reference line through the center of attraction O. The position of a point P on the orbit will be determined by its polar coordinates (r,θ), θ being measured in the same manner as ω.

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Thus, the polar equation of the orbit can be written in the familiar form

$$l/r = 1 + e\cos(\theta - \omega) \tag{2}$$

Suppose a rocket moving in the latter orbit at the point P and having acceleration γ/r^2 toward O is subjected to an impulsive thrust I at an angle ϕ to the forward perpendicular to OP. Then the elements (l,e,ω) of its new orbit must continue to satisfy Eq. (2). However, it is proved in Ref. 2 that these new elements will also satisfy the equation

$$e\sin(\theta - \omega) = (l/r - l^{1/2}Z)\tan\phi \tag{3}$$

where $\gamma^{1/2}Z\sin\phi$ represents the rocket's velocity component perpendicular to the impulse that is unaffected by the thrust. Thus, for given values of r, θ, Z, ϕ, the elements (l,e,ω) of all possible orbits that can be entered by application of an impulse in the direction ϕ must satisfy Eqs. (2) and (3).

If (l,e,ω) and (l',e',ω') are two sets of elements satisfying Eqs. (2) and (3), then it is also proved in Ref. 2 that the velocity increment required to move the vehicle from the first orbit to the second is given by

$$\Delta V = \gamma^{1/2}(l'^{1/2} - l^{1/2})/(r\cos\phi) \tag{4}$$

$f = \theta - \omega$ is the real anomaly. If E is the eccentric anomaly, a the semimajor axis, and t the time of transit from the periapse, the following equations are well known:

$$\cos E = \frac{e + \cos f}{1 + e\cos f}, \qquad \cos f = \frac{\cos E - e}{1 - e\cos E} \tag{5}$$

$$a = l/(1 - e^2), \qquad r = a(1 - e\cos E) \tag{6}$$

$$t = a^{3/2}(E - e\sin E)/\gamma^{1/2} \tag{7}$$

Throughout the Note, all orbits are assumed to be described in the positive sense.

Primer Vector on Keplerian Arcs

The problem of the transfer of a rocket between a pair of fixed terminals (at which its velocity is known) in a given gravitational field with a minimum expenditure of propellant has been solved in Ref. 1 in terms of a primer vector defined along the trajectory. If we assume the absence of arcs of intermediate thrust (IT arcs), the optimal trajectory is known to comprise coasting arcs separated by junction points where impulsive thrusts are applied. For optimality, the magnitude of the primer must assume a maximum value of unity at all junctions and, at these points, the primer and thrust vectors must be aligned. Also, across each junction, the primer and its first derivative must be continuous.

In the case where the motion is confined to a plane and the field follows an inverse square law, along an orbit having elements (l,e,ω), the radial and transverse components (λ,μ) of the primer p have been shown to be given by

$$\lambda = A\cos f + Be\sin f + (l^2/e)CI\sin f \tag{8}$$

$$\mu = -A\sin f + B(1+e\cos f) + \frac{D - A\sin f}{1+e\cos f} + \frac{l^2}{e^2}C\left[\frac{\cot f}{1+e\cos f} + (1+e\cos f)I\right] \tag{9}$$

where $f = \theta - \omega$, (A,B,C,D) are constants, and

$$I = \int \frac{\mathrm{d}f}{\sin^2 f(1+e\cos f)^2} \tag{10}$$

In general, the values of the constants (A,B,D) change each time the rocket is transferred to a new Keplerian arc at a junction. The constant C, however, arises from a first integral of the optimality equations and keeps the same value over the whole trajectory. This first integral is exhibited as Eq. (5.39) in Ref. 1. In the subsequent equation (5.40), a factor (l/γ) has been absorbed into C. This is not permissible, since the element l changes as we move from one arc to another, whereas C remains invariant. Further, additional factors $1/e$ and l have been absorbed in Eqs. (5.49) and (5.55–5.56), respectively. Thus, it is necessary to replace C in Eqs. (5.55) and (5.56) by l^2C/e; these equations then become identical with the foregoing Eqs. (8) and (9) (the factor $1/\gamma$ can, of course, be absorbed without error). Subsequent calculations in Ref. 1 are not invalidated, since only the case $C = 0$ (the time-open case) has there been analyzed.

The radial and transverse components (ξ,η) of the time derivative $\dot{p}$ of the primer are given in Eqs. (5.63) and (5.64) of Ref. 1. After correction of the constant C and inclusion of a missing index, these take the form

$$\xi = \frac{\sqrt{(\gamma l)}}{r^2}\left\{\frac{A\sin f - D}{1+e\cos f} - B + \frac{l^2}{e^2}C\left[\frac{e\sin^2 f - \cos f}{\sin f(1+e\cos f)^2} - I\right]\right\} \tag{11}$$

$$\eta = \frac{\sqrt{(\gamma l)}}{l^2}\left[-A(e+\cos f) + De\sin f + \frac{l^2}{e}C\cos f\right] \tag{12}$$

The integral I is most easily evaluated in terms of the eccentric anomaly E. Referring to Eqs. (5), we can show that

$$\begin{aligned} I &= \int \frac{\mathrm{d}f}{\sin^2 f(1+e\cos f)^2} \\ &= (1-e^2)^{-5/2}\int (1-e\cos E)^3 \operatorname{cosec}^2 E\,\mathrm{d}E \\ &= (1-e^2)^{-5/2}\big[-(1+3e^2)\cot E + (3e+e^3)\operatorname{cosec} E \\ &\quad - 2e^3\sin E - 3e^2(E - e\sin E)\big] \\ &= \frac{1}{1-e^2}\left[\frac{2e}{\sin f(1+e\cos f)} - \cot f - \frac{3\gamma^{1/2}}{l^{3/2}}e^2 t\right] \end{aligned} \tag{13}$$

where t is the orbital time from the periapse given by Eq. (7).

Necessary Conditions for Optimality

Suppose that, at the first junction J, the impulsive thrust transfers the rocket from the orbit (l_1,e_1,ω_1) to the orbit (l,e,ω). Then, if (r_1,θ_1) are the polar coordinates of J, Eqs. (2) and (3) show that we must have

$$e_1\cos(\theta_1 - \omega_1) = l_1 s_1 - 1, \qquad e\cos(\theta_1 - \omega) = l s_1 - 1 \tag{14}$$

$$e_1\sin(\theta_1 - \omega_1) = (l_1 s_1 - l_1^{1/2} Z_1)\tan\phi_1$$

$$e\sin(\theta_1 - \omega) = (l s_1 - l^{1/2} Z_1)\tan\phi_1 \tag{15}$$

where Z_1, ϕ_1 are the values of Z, ϕ appropriate to the impulse at J and $s_1 = 1/r_1$.

On both the Keplerian arcs meeting at J, the components (λ,μ) of the primer must take values $(\sin\phi_1, \cos\phi_1)$ at the junction. Referring to Eqs. (8) and (9), we see that these conditions require that

$$\begin{aligned} &A_1\cos f_1 + B_1 e_1\sin f_1 + \frac{l_1^2}{e_1}CI_1\sin f_1 = A\cos f \\ &+ Be\sin f + \frac{l^2}{e}CI\sin f = \sin\phi_1 \end{aligned} \tag{16}$$

$$\begin{aligned} &-A_1\sin f_1 + B_1(1+e_1\cos f_1) + \frac{D_1 - A_1\sin f_1}{1+e_1\cos f_1} \\ &+ \frac{l_1^2}{e_1^2}C\left[\frac{\cot f_1}{1+e_1\cos f_1} + (1+e_1\cos f_1)I_1\right] \\ &= -A\sin f + B(1+e\cos f) + \frac{D - A\sin f}{1+e\cos f} \\ &+ \frac{l^2}{e^2}C\left[\frac{\cot f}{1+e\cos f} + (1+e\cos f)I\right] = \cos\phi_1 \end{aligned} \tag{17}$$

where $f_1 = \theta_1 - \omega_1$, $f = \theta_1 - \omega$, and I is given by Eq. (13) (I_1 is obtained from I by replacing e, f, l, t by e_1, f_1, l_1, t_1).

We derive two further equations from the requirement that $\dot{p}$ should be continuous across J. Referring to Eqs. (11) and (12), we express this condition by the equations

$$\begin{aligned} &\frac{\sqrt{(\gamma l_1)}}{r_1^2}\left\{\frac{A_1\sin f_1 - D_1}{1+e_1\cos f_1} - B_1 \right.\\ &\left. + \frac{l_1^2}{e_1^2}C\left[\frac{e_1\sin^2 f_1 - \cos f_1}{\sin f_1(1+e_1\cos f_1)^2} - I_1\right]\right\} \\ &= \frac{\sqrt{(\gamma l)}}{r_1^2}\left\{\left[\frac{A\sin f - D}{1+e\cos f} - B \right.\right.\\ &\left.\left. + \frac{l^2}{e^2}C\left[\frac{e\sin^2 f - \cos f}{\sin f(1+e\cos f)^2} - I\right]\right]\right\} \end{aligned} \tag{18}$$

$$\begin{aligned} &\frac{\sqrt{(\gamma l_1)}}{l_1^2}\left[-A_1(e_1+\cos f_1) + D_1 e_1\sin f_1 + \frac{l_1^2}{e_1}C\cos f_1\right] \\ &= \frac{\sqrt{(\gamma l)}}{l^2}\left[-A(e+\cos f) + De\sin f + \frac{l^2}{e}C\cos f\right] \end{aligned} \tag{19}$$

The six equations (16–19) can now be solved to determine the six parameters A, B, D, A_1, B_1, D_1. This is a straightforward, but laborious, process and leads to results that may be expressed [using Eqs. (14) and (15)] in the form

$$eA/l^{1/2} = (Z_1 - s_1/Z_1)\sin\phi_1 + C/(s_1 Z_1) \tag{20}$$

$$\begin{aligned} eB &= \left[1 + 1/(l^{1/2}Z_1)\right]\cos f\cos\phi_1 + \sin f\sin\phi_1 \\ &\quad - \frac{l^2C}{e}\left(\frac{\cot f}{l^{3/2}s_1 Z_1} + I\right) \end{aligned} \tag{21}$$

$$\begin{aligned} B + D &= -\cos\phi_1\left(1 + \frac{ls_1 + 1}{l^{1/2}Z_1}\right) \\ &\quad - \frac{l^2C}{e^2}\left[\cot f - \frac{e(ls_1+1)}{l^{3/2}s_1 Z_1\sin f} + I\right] \end{aligned} \tag{22}$$

with corresponding equations for A_1, B_1, D_1 (replace e, l, f, I by e_1, l_1, f_1, I_1, respectively).

The reader wishing to check these formulas is advised to verify them by direct substitution into Eqs. (16-19). Judicious use needs to be made of the relationship

$$e^2 = (ls_1 - 1)^2 + (ls_1 - l^{1/2}Z_1)^2\tan^2\phi_1 \tag{23}$$

which follows from Eqs. (14) and (15). It will then be found that the right-hand members of Eqs. (18) and (19) yield the values of the components (ξ,η) of $\dot{p}$ at the junction in the form

$$\xi = -\frac{\gamma^{1/2}}{Z_1}\cot\phi_1(C - s_1^2\sin\phi_1) \tag{24}$$

$$\eta = \frac{\gamma^{1/2}}{Z_1}(C - s_1^2\sin\phi_1) \tag{25}$$

We note immediately that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(p^2) = p\cdot\dot{p} = \lambda\xi + \mu\eta = \xi\sin\phi_1 + \eta\cos\phi_1 = 0 \tag{26}$$

showing that the magnitude of the primer is stationary at the junction. It was proved in Ref. 1 (p. 64) that this is a consequence of the constancy of C over the whole trajectory.

More precisely, it is necessary that p be a maximum at the junction. A necessary condition for this to be so is

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(p^2) = p\cdot\ddot{p} + \dot{p}^2 \le 0 \tag{27}$$

To apply this condition, we need the components of $\ddot{p}$ at the junction. Referring to Ref. 1 (p. 81), we find that the radial and transverse components of $\ddot{p}$ at any point on a Keplerian arc are $\gamma r^{-3}(2\lambda, -\mu)$. It follows that condition (27) requires that

$$(\gamma/r_1^3)(2\sin^2\phi_1 - \cos^2\phi_1) + \xi^2 + \eta^2 \le 0 \tag{28}$$

which reduces to

$$|Z_1| \ge \frac{|Cr_1^2 - \sin\phi_1|}{r_1^{1/2}|\sin\phi_1|\sqrt{(1-3\sin^2\phi_1)}} \tag{29}$$

One consequence of this condition is clearly that $|\sin\phi_1| < 1/\sqrt{3}$, a well-known condition. For the problem of time-open optimal transfer, we have $C=0$ and the condition (29) reduces to

$$\frac{1}{|Z_1|} \le r_1^{1/2}\sqrt{(1-3\sin^2\phi_1)} \tag{30}$$

This is a new condition that needs to be satisfied at all junctions on an optimal trajectory. Since $w_1 = \gamma^{1/2}|Z_1\sin\phi_1|$ represents the magnitude of the velocity component of the rocket that is unaffected by the impulse and $V_c = \sqrt{(\gamma/r_1)}$ is the circular velocity at the junction, the condition (29) can be expressed in the form

$$w_1 \ge \frac{|Cr_1^2 - \sin\phi_1|}{\sqrt{(1-3\sin^2\phi_1)}}V_c \tag{31}$$

This condition must be satisfied at both junctions.

Finally, we can now write down the primary conditions to be satisfied by an optimal two-impulse transfer by requiring that the values of A, B, D, derived at the junction J, are the same as those derived at the junction K.

It will be convenient first to specify three expressions involving parameters that are defined at both junctions. Thus,

$$L = (Z - s/Z)\sin\phi - e\alpha/(l^2sZ) \tag{32}$$

$$M = \cos(\psi-\phi) + \frac{1}{l^{1/2}Z}\cos\psi\cos\phi + \alpha\left[\frac{\cot\psi}{l^{3/2}sZ} + \frac{1}{1-e^2}\left(\frac{2e}{ls\sin\psi} - \cot\psi\right)\right] \tag{33}$$

$$N = \cos\phi\left(1 + \frac{ls+1}{l^{1/2}Z}\right) + \alpha\left[\frac{ls+1}{l^{3/2}sZ\sin\psi} + \frac{1}{1-e^2}\left(e\cot\psi - \frac{2}{ls\sin\psi}\right)\right] \tag{34}$$

where $\psi = \theta - \omega$, $\alpha = -l^2C/e$. Then, (L_1, M_1, N_1) will denote the same quantities when (s,θ,Z,ϕ) are replaced by $(s_1,\theta_1,Z_1,\phi_1)$. (L_2, M_2, N_2) are defined at the other junction similarly [note that the elements (l,e,ω) have the same values at both junctions]. The necessary conditions for optimality can then be written in the form

$$L_1 = L_2,\quad M_1 = M_2 - \frac{3\gamma^{1/2}\alpha e^2\tau}{l^{3/2}(1-e^2)},\quad N_1 = N_2 + \frac{3\gamma^{1/2}\alpha e\tau}{l^{3/2}(1-e^2)} \tag{35}$$

where τ denotes the time in the transfer orbit JK.

Further applicable equations are the counterparts of Eqs. (14) and (15) for the junction K, viz.,

$$e_2\cos(\theta_2-\omega_2) = l_2s_2 - 1,\qquad e\cos(\theta_2-\omega) = ls_2 - 1 \tag{36}$$

$$e_2\sin(\theta_2-\omega_2) = (l_2s_2 - l_2^{1/2}Z_2)\tan\phi_2$$

$$e\sin(\theta_2-\omega) = (ls_2 - l^{1/2}Z_2)\tan\phi_2 \tag{37}$$

Finally, there is the constraint on the total time of transfer between the terminals, viz., Eq. (1). t_1, t_2, and τ are calculable from Kepler's equation (7).

Equations (14), (15), (35–37) and the constraint (1) now provide 12 conditions determining the 12 unknowns s_1, θ_1, Z_1, ϕ_1, s_2, θ_2, Z_2, ϕ_2, l, e, ω, α. Having calculated these quantities, we see that the parameters at each junction must further satisfy the inequality (29) if the characteristic velocity for the maneuver [as determined by Eq. (4)] is to be a local minimum.

Transfers Using More Than Two Impulses

It can happen that no solution to our equations can be found to satisfy the inequality (29) at both junctions J and K. In such circumstances, a two-impulse maneuver cannot be even locally optimal and transfers using three or more impulses need consideration.

If we suppose there to be three junction points J, K, L, then the necessary conditions (35) will apply to the junctions J and K, and a similar triad of conditions can be written down in respect to the junctions K and L (l,e,ω now refer to the Keplerian arc KL). The elements of the arc KL constitute three additional unknowns, but the three additional equations just mentioned are now available for their determination. However, a numerical search for solutions would be difficult and it would probably be preferable to improve the two-impulse solution by the iterative procedure of Lion and Handelsman,[7] as has been done for the case when the terminal orbits are both circular by Prussing and Chiu.[6]

Conclusion

Previously published necessary conditions governing the optimal transfer between a pair of coplanar elliptical orbits using two impulses have been extended to the case when the time of transfer is prescribed. A new inequality condition to be satisfied at all junctions has also been found.

References

[1]Lawden, D. F., *Optimal Trajectories for Space Navigation,* Butterworths, London, 1963.

[2]Lawden, D. F., "Impulsive Transfer Between Elliptical Orbits," *Optimization Techniques,* edited by G. Leitmann, Academic Press, New York, 1962, pp. 323–351.

[3]Lawden, D. F., "Optimal Transfers Between Coplanar Elliptical Orbits," *Journal of Guidance, Control, and Dynamics,* Vol. 15, No. 3, 1992, pp. 788–791.

[4]Prussing, J. E., "Optimal Four-Impulse Fixed-Time Rendezvous in the Vicinity of a Circular Orbit," *AIAA Journal,* Vol. 7, No. 5, 1969, pp. 928–935.

[5]Prussing, J. E., "Optimal Two- and Three-Impulse Fixed-Time Rendezvous in the Vicinity of a Circular Orbit," *AIAA Journal,* Vol. 8, No. 7, 1970, pp. 1221–1228.

[6]Prussing, J. E., and Chiu, J. H., "Optimal Multiple-Impulse Time-Fixed Rendezvous Between Circular Orbits," *Journal of Guidance, Control, and Dynamics,* Vol. 9, No. 1, 1986, pp. 17–22.

[7]Lion, P. M., and Handelsman, M., "The Primer Vector on Fixed-Time Impulsive Trajectories," *AIAA Journal,* Vol. 6, No. 1, 1968, pp. 127–132.